

Oliver Heaviside, Maxwell's Apostle and Maxwellian Apostate

by

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Yoy may, if you like & with some sense, say it is not Maxwell, but is Heaviside. Nevertheless I am sure he would have accepted the change at once when pointed out to him, & have adopted it in his own work. [Heaviside to Lodge: Oct. 30, 1893. UCL. Lodge Collection]

I. Background

Oliver Heaviside was self-educated, eccentric, willful, isolated, suspicious and brilliant. He rightly considered himself to be the guardian of the true spirit of Maxwell's theory, which he defended with a mixture of novel symbolism, impenetrably brief analysis and biting sarcasm. "Any account I could give of him", wrote George Searle, one of Heaviside's few close acquaintances, "must seem odd to the uninitiated, for the simple reason that all I ever saw of him was odd". [Searle, 1950]

Heaviside's oddness is amply apparent in his letters and even in his published work. In later years his eccentricity was greatly exacerbated by his not having had a Cambridge background, though almost everyone competent to understand him had attended the university. Yet Heaviside's informal education permitted him to create a version of electrodynamics which captured the essence of British dynamical reasoning while avoiding the highly abstract methods (including Hamilton's principle and Lagrange's equations) in common use among Cambridge graduates.

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We know little about Heaviside's first thirty years. He was born in London in 1850 and probably lived in the house of his birth until sometime between 1862 and 1871. In 1876 the family moved to their third house since his birth – still in the same part of London. Indeed, Oliver, the youngest of four sons, lived for the most part with his parents until their deaths in the mid-1890s.

Heaviside had very little formal education, though he was the nephew on his mother's side of Sir Charles Wheatstone, whose influence seems to have been limited to encouraging Heaviside and his brother Arthur to pursue careers in telegraphy. Heaviside's elementary teacher at the Camden House School was a Mr. Cheshire, whom he evidently greatly admired [Lee, 1950: 10–11], but about whom we know nothing. In 1865 Heaviside took the College of Preceptors examination in thirteen subjects. He obtained 1140 marks out of 2600, which put him fifth overall and first in the Natural Sciences. He did not go on to university but, apparently on his own, studied "Danish, German, and the Natural Sciences". [Lee, 1950: 10–11] This enabled him to take up a position as a telegraph operator in 1868 with the *Dansk-Norsk-Engelske Telegraf Selskab*, for whom he performed tests on signalling speed along the newly-laid cable between Newcastle-on-Tyne and Fredericia. He returned to England in 1870 when the Telegraf was absorbed by the Great Northern Telegraph, for whom he turned home to London to live with his parents. Heaviside never thereafter held a job and often suffered from poverty, brought on in some measure by his persistent refusal to accept the help of the few colleagues with whom he maintained contact.¹

The only detailed biography of Heaviside, though brief, was written in 1929 shortly after his death. We may take it that Appleyard's descriptions of Heaviside at least in part reflect the experiences of those who knew him. We read of his daily routine that:

... from 1876 Oliver subjected himself to strenuous training. His habit was to retire to his room at about 10 o'clock at night and to work there until the early hours of the morning. He closed his door and windows, lighted his oil lamp, and allowed the air to become hot and stifling. He worked also during the day, in seclusion. In order that he might not be disturbed, his food was placed outside his door, and there it remained until he was disposed to take it. [Appleyard, 1929: 215]

Heaviside was also somewhat deaf, which often made him suspicious, and he lacked almost all sense of smell. The picture emerges of a sharp-minded auto-didact who never married, lived with his parents, and remained intellectually isolated for most of his life.

By the early 1870s Heaviside was collaborating with one of his brothers (Arthur West Heaviside, a Divisional Engineer with the British General Post Office) in experiments “on duplex working with an artificial line”. [Appleyard, 1929: 221, quoting from a memorandum dated Jan. 15, 1873] He began a series of publications on signalling through cables in 1872, and this subject, with an increasingly complex approach, occupied most of his attention until the mid-1880s, brought him reknown, and in many ways influenced his understanding of Maxwell’s theory.

Heaviside’s earliest publications show his mathematical originality and his clear grasp of basic electromagnetic principles. Nevertheless this work does not provide much direct evidence concerning the sources of his knowledge. However, in an important article of 1885 which I shall discuss below, Heaviside remarked:

It so happened that my first acquaintance with electricity was with the dynamic phenomena, and after I had read with absorbed interest that instructive book, Tyndall’s ‘Heat as a Mode of Motion’. This may explain why, when it came later to book-learning regarding electricity, I had the greatest possible repugnance to all the explanations, and could not accept the electric current to be the motion of electricity (static) through a wire, but thought it something quite different. I simply did not believe, except so far as mere statements of experimental facts were concerned. [Heaviside, 1892 (1885a): 435]

Tyndall’s widely-read *Heat as a Mode of Motion* consists of simple but evocative illustrations of energy conservation and conversion. The underlying physical imagery assumes ‘atoms’ acted on by attractive and repulsive forces, heat being ‘atomic’ motion. Perhaps the most striking characteristic of the book was its constant emphasis upon the role of energy transformations in all physical processes:

To create or annihilate energy is as impossible as to create or annihilate matter; and all the phenomena of the universe consist in transformations of energy alone. [Tyndall, 1875: sec. 155]

Indeed, Tyndall’s *Heat* focuses almost exclusively upon concrete cases which illustrate graphically the proposition that energy changes in

form but never in amount. This book apparently had an important effect on the young Heaviside. He, like Tyndall, always emphasized the role of energy transformations. One might even say, without undue exaggeration, that the essence of a theory was contained for Heaviside in its expression as a means for transforming energy. This is strikingly evident in his first published work on Maxwell's theory.

II. Maxwell's Theory and Dynamics

We do not know precisely what prepared Heaviside for his first reading of Maxwell, which occurred in 1873 when he obtained a copy of the *Treatise* shortly after its publication. [Lee, 1950: 11] However, in 1888 he wrote Oliver Lodge a letter to protest Rayleigh's characterization of him (Heaviside) as somehow anti-Maxwell which provides some information:

Dear Professor Lodge,

There is a passage in Lord R's letter (which I return) which at once surprised & rather pained me: – "No one would guess from the tone in which he writes of Maxwell that he had learned nearly all he knows from M." If this be really the case, it shows how thoroughly one may be misunderstood from some chance remark which is only a small part of what one has said. As you may share in the misunderstanding, I will state how the case really stands. On the basis of a careful study of the best part of Thomson's electrical writings, and of Maxwell's *Treatise*, and of the views of some of the German speculators (which I detest, although I gave great attention to them) I became convinced that M. was a "heaven-born" genius, as he has been called, & that his system was that of the future; his work the *Principia*. And from '83 to '87 I have, in the *Electrician* endeavoured to popularise and extend M's way of looking at things electrical, or rather, what I fancied it would be if M. had given his own special views fairer play than he did, & made them prominent always. My work has always been professedly an exposition of his views, with some developments, & I have repeatedly referred admiringly to him. [Heaviside to Lodge, Oct. 9, 1888, UCL Lodge Collection]

Having studied William Thomson, whose own work early exemplified an emphasis upon energy formulations,² Heaviside read Maxwell's *Treatise*. The experience affected him deeply. In the *Treatise* he found a theory, albeit one often hidden, in which energy considerations play a central role. In Maxwell's electrodynamics all forces – whether internal to the field proper or acting between the field and matter – involve

localised energy changes because force in the deepest sense must, for Maxwell, reflect an energy gradient at the locus of action.³

This concept deepens and extends Tyndall's broad emphasis upon energy transformations; it embodies the essence of the requirement that physical processes always involve energy conversions in the mathematical image of an energy function defined at each point of space. In such a scheme 'force' appears as an implication of an inhomogeneous configuration of energy which, when the configuration alters (i.e. the 'force' acts), involves energy transformation.

These concepts revealed to Heaviside the necessity of treating the world as a continuum. Only if one can assume continuous energy functions can actions be fully reduced to specifications of energy gradients. This was likely a novel idea for Heaviside given his early reading of Tyndall's *Heat*, which continually invokes the image of forces acting at a distance between atoms distributed in the void. In such a system it makes little sense to think of continuous energy distributions – indeed, it is even difficult to envision where energy that is not kinetic can be stored. Yet the idea of a continuous energy distribution is built deeply into the heart of Maxwell's *Treatise* and cannot be missed by an attentive reader. What evidently impressed Heaviside about Tyndall's *Heat* was its focus upon energy rather than the physical suppositions Tyndall used to justify the focus.

We can see clearly the central role played by localised energy distributions (and hence by the necessary assumption of continuum analysis) in Heaviside's earliest publication on Maxwell's theory, which concerned the energy of the electric current. [Heaviside, 1892 (1883)] Here Heaviside defined force proper in relation to energy, writing of "‘force’ in the generalised sense, so that ‘force’ $\times$ ‘displacement’ always equals the decrease of potential energy during the displacement". [Heaviside, 1892 (1883): 234]

This emphasis upon energy, rather than force, as a foundation led directly to Heaviside's initial disquiet with Maxwell's vector potential (A). [A point remarked by Jackson, 1950: 57] Though Heaviside began his analysis of current energy by expressing the energy of a current C in the form $\frac{1}{2} \int A \cdot C d^3x$, he soon turned to the equivalent expression $\int (B^2/8\pi) d^3x$, integrating over all space, where B is magnetic induction. In the form involving A the energy distribution cannot reasonably be assigned a volume density $\frac{1}{2} A \cdot C$ since there would then

be no energy in regions devoid of current – and so no possibility of energy analysis for these regions. But in the alternate form we can assign a universal density $B^2/8\pi$. That is, the vector potential appears to be unsatisfactory as a physical entity because it forbids analysis based on localized energy distributions, and these latter are essential if we are to base our theory directly upon energy transformations as Heaviside wished.

In 1885 Heaviside made his commitment to an energy-based physics even more explicit, going so far as to discuss ‘dynamics’ in terms which at once invoke energy principles:

All the physical sciences are bound to become branches of dynamics in course of time, and anything contradicting the principles of dynamics should be unhesitatingly rejected. Without having made an exhaustive study of dynamics, I have yet managed to come to the conclusion that a force cannot act where it is not – meaning by acting, the doing of work ... All working forces involve transfer of energy, and the measure of a force, whether simple or generalised, is the amount transformed per unit change of the variable to which the force corresponds. [Heaviside, 1892 (1883): 247]

On first encountering Heaviside’s ideas on force, energy, and dynamics one might find little that differs substantially from the contemporary mainstream of Cambridge-trained physicists. His ideas do have much in common with those of the mainstream, as one would naturally expect given that, by the early 1890s, Heaviside was widely thought of as an insightful, if obscure, analyst of Maxwell’s theory. British physicists did share with Heaviside a firmly-seated conviction that forces inevitably reflect energy transitions which occur locally in a continuum. But unlike Heaviside they sought to embed that idea, indeed even to derive it, from the intricate structure of Hamilton’s principle and Lagrange’s equations.

To Heaviside’s British contemporaries the deeper reality was abstract dynamics. For Heaviside it was quite the other way around. Whatever usefulness abstract dynamics had derived entirely from its fit with energy processes known on independent grounds to occur in certain ways. Lagrange’s equations, and Hamilton’s principle, were in Heaviside’s eyes useless extravagances. In later years he was even moved to lash out vehemently at the widespread use of these methods. He wrote in 1900:

It is ... striking that the method of treating dynamics in use by mathematicians in general, namely, the method of generalized coordinates and its complications, does not usually show up well in the treatment of electromagnetics. I have observed repeatedly that young to-be physicists, when they leave college, are full of generalized coordinates, and the theory of functions, and unnatural spaces, and rigour; but when they become thoroughly immersed in real physics, even though mathematical, a lot of the learning referred to fades away. Is Cambridge to blame? Perhaps not. It may be my fault. [Heaviside, 1892 (1885b): 419]

Heaviside's profound dislike of the analytically complex dynamics practiced by Cambridge men became intense in later years, but it was present in embryo from the beginning. It derives in the end from his strongly held conviction that mathematics which does not provide direct insight into the play of energy characteristic of some process necessarily obscures its physical essence. Yet the method of "generalized coordinates" (Lagrange's equations) which Heaviside so disliked was in many respects a foundational one in contemporary British physics.

How then could Heaviside do without these distasteful techniques? He did so by avoiding the situations where they seemed to be essential and by providing an alternative method where they might otherwise have been used. For example, the three phenomena which stimulated the most intense use of the methods – the Faraday, Kerr and Hall effects – were almost entirely ignored by Heaviside, except for an occasional remark in his correspondence. Nor did Heaviside ever discuss in any detail the kinds of processes (e.g. electric and magnetic effects attendant of thermal or mechanical actions) which J. J. Thomson attacked in detail with Lagrange's equations in the mid 1880s. [See Buchwald, 1985: Part II] Heaviside ignored almost every phenomenon, such as these, in which the pure field equations had to be altered and new constants introduced. Whereas his contemporaries focused much on their attention on these kinds of situations, Heaviside serenely paid them no heed. He was preeminently concerned with applying Maxwell's theory to situations in which only elementary properties – specific inductive capacity, magnetic permeability and conductivity – figured.

Heaviside's preference for a purified Maxwell theory based directly upon energy considerations and untainted by unusual phenomena required him to do two things. First, he had to restructure Maxwell's theory in ways that lent themselves to direct interpretations using con-

tinuous energy functions. Second, he needed a method for expressing energy transformations which avoided Lagrange's equations. He provided both of these necessities as early as 1885, but the second – his alternative to the Lagrange equations – was only fully developed in 1891.

III. The Reformulation of Maxwell's theory

Heaviside's version of Maxwell's theory began to appear in print in 1885. Though his account is entirely coherent, nevertheless it has about it a certain diffuseness common to much of his published work. New concepts appear, are partially discussed and then disappear only to reemerge in a different way pages or even years later. Yet close examination of Heaviside's work after 1885 shows little fundamental change. Concepts introduced earlier are pursued in greater detail in subsequent publications, but rarely if ever do we find any conflict with previous assertions. Consequently we may date the emergence of Heaviside's overall approach to no later than 1884–5.

Heaviside shared with Maxwellians the essential concept that all electromagnetic phenomena reflect field processes. In particular, he always considered 'charge' to be due to processes occurring in regions where the ratio of specific inductive capacity to conductivity varies from point to point. That understanding of 'charge', together with an emphasis upon the play of energy in the continuum, form two basic elements of all Maxwellian theories. Yet even in his treatment of 'charge' Heaviside diverged from his contemporaries, as we shall see – though he diverged more in his analytical method than in his basic understanding.

We must begin with Heaviside's reformulation of Maxwell's equations on a basis suited to express energy properties of the field. Underlying this reformulation is the concept of "impressed force", which remained central to Heaviside's work. His first clarification of the idea occurred in 1884 in his treatment of the energy of currents. He wrote then:

The most general definition of impressed force is that depending on energy, as, in fact, all forces, simple or generalised, are expressed by energy variations. If $\bar{e}$ be the im-

pressed force within a certain unit volume of the conductor, and there be a current of density $\vec{C}$ at that place, and in the same direction as $\vec{e}$, then the amount of energy taken in per second within that unit volume by the current is represented by the product $\vec{e}[\cdot]\vec{C}$... [Heaviside, 1892 (1883): 349–350]

This partial illustration was meant to introduce impressed force as an external action which feeds energy into the system, as Heaviside shortly made clear:

In an isolated, conservative system, all transfers of energy are internal, and its total energy remains constant. If such a system receives energy, this involves impressed force, and a system communicating the energy. What to consider as impressed force depends upon how large we make our system, and is therefore considerably a matter of choice. [Heaviside, 1892 (1883): 349–50]

Using the concept of impressed force Heaviside reformulated Maxwell's equations in a way that brings out quite directly their link to energy considerations. Compressing his discursive account, we have in the end a compact set of equations. Begin by introducing four field vectors: e , E , and h , H . The upper-case vectors E , H respectively represent the complete electric and magnetic fields in a system, including impressed as well as internal contributions. The lower-case vectors e , h represent the impressed forces, and Heaviside assumed that they must be specified independently. In his reformulation the curls of the non-impressed parts of E , H yield, respectively, what he termed magnetic and electric "currents". The scalar products of the impressed parts with these "currents" determine the rates at which energy is fed into the system. To maintain a strict energetic foundation Heaviside, we shall see, introduced the (empirically non-existent) coefficient, g , of magnetic conductivity.

Heaviside began with "Ohm's law": [Heaviside, 1892 (1885a): 419]

$$C = kE. \quad (1)$$

He turned next to energy considerations and introduced what he termed the "dissipativity", Q – the volume density of the rate at which energy dissipates in the conduction current:

$$Q = E \cdot C. \quad (2)$$

In an important locution which he used extensively in later years, Heaviside termed Q the “rate of working of the force E ”.

Having begun with the conduction current and the dissipation which accompanies it, Heaviside next introduced the electric displacement, D :

$$D = \frac{1}{4\pi} \epsilon E. \quad (3)$$

He used (3) to express the energy density of the electric field:

$$U = \frac{1}{2} E \cdot D. \quad (4)$$

In Heaviside’s energy-based formulation fields may generate “currents” and necessarily expend energy in so doing: in his understanding fields act by causing energy transformations, and to do so they must ‘work’ upon something. In the case of equation (2) E ‘works’ on the conduction current C . But E could also ‘work’ upon the displacement current, though the energy obtained as a result of this ‘working’ would not be dissipated but stored reversibly in the displacement. Heaviside required that the rate at which a field acts on its correlate ‘current’ must always be given by the scalar product of the field by the ‘current’. (If a rate of working conflicted with other propositions known independently to be true then the implication was that the expression used in it for the current had to be wrong.) The field force, E , then, ‘works’ on a current, J , which Heaviside assumed to consist of the sum of C – the conduction current – and $\partial D/\partial t$ – the displacement current:

$$J = C + \frac{\partial D}{\partial t}. \quad (5)$$

Consequently the “rate of working” of E is $E \cdot J$ or, in virtue of (2)–(5), just $Q + \partial U/\partial t$.

Heaviside extended this approach to the magnetic field, and this impelled him to introduce a coefficient, g , of magnetic conductivity in precise energetic parallel to the coefficient, k , of electric conductivity.

He began by introducing B , the magnetic induction, and T , the energy density in the magnetic field:

$$B = \mu H \quad (6)$$

$$T = \frac{1}{8\pi} H \cdot B. \quad (7)$$

The force field, H , has, like E , a ‘current’ upon which it works. Heaviside denoted this ‘current’ G and defined it as the sum of an induction current and a ‘current’ of magnetic conduction (assumed to satisfy a magnetic Ohm’s law): [Heaviside, 1892 (1885b): 429 & ff.]

$$G = \frac{1}{4\pi} \frac{\partial B}{\partial t} + gH. \quad (8)$$

Then we have $H \cdot G$ for the “rate of working” of H upon G , and by (6)–(7) this is:

$$H \cdot G = H \cdot gH + \frac{\partial T}{\partial t}. \quad (9)$$

The presence of g requires that Q , the dissipativity, must be redefined to include the effect of magnetic conduction:

$$Q = E \cdot C + H \cdot gH. \quad (2')$$

Though Heaviside at once remarked that “there is probably no such thing as a magnetic conduction current”, he spent the following two pages examining what would occur in a special case if it did exist. Nevertheless we must be careful not to attribute too much to the role of g . It is not a question of the actual existence of magnetic conductivity. Whether or not g is always zero in fact is an experimental question. What was important for Heaviside was the dynamical structure of the equations. For completeness one must include magnetic conductivity else there is a rate of electric working ($E \cdot C$) for which there is not even a formal magnetic analog. In a theory founded upon the working

of force fields (E, H) upon 'currents' (J, G) a lack of dynamical symmetry would be intolerable. It may very well be that g is always and everywhere zero, but we must build our theory without prejudging the issue. For Heaviside 'facts' – here the existence of g – must not be allowed to disturb unduly an otherwise well-founded theory. He elsewhere remarked:

Certainly theory must ultimately be made to agree with facts; but when some few facts do not apparently fit into a theory which suits a much greater number of facts, it becomes a question of balance of advantages whether it would be better to alter theoretical notions, or to leave the facts unexplained for the time, waiting for further information, or for new light on the question of fitting the facts into the theory. [Heaviside, 1892 (1885b): 441]

If magnetic conductivity does not seem to exist, we must nevertheless allow its presence in our theory in order to preserve dynamical completeness – and later we may explain the peculiar fact that g is ever zero.

Heaviside's introduction of magnetic conductivity, more than any other of his several novelties, reflects his adherence to the deepest elements of Maxwell's theory. For g permits more than a complete and symmetric formulation in terms of energy. It allows as well the existence of magnetic 'charge' on precisely the same kinds of grounds that allow for electric 'charge': here in circumstances wherein the ratio of magnetic permeability (μ) to magnetic conductivity (g) varies from point to point. Heaviside seems almost to have felt that g should exist even if it does not. The oracular character of much of his writing is in part due to his difficulty in expressing the deep-seated image which he had of how the world ought to work. He felt viscerally the constant flux of fields waxing, waning and subsiding, continuously transforming and transferring energy. Magnetic conductivity represented the possibility of magnetic subsidence, a mode of transformation which, in a well-ordered world, would be possible for all fields.

What we are thusfar missing, of course, are the Ampère and Faraday laws. These appear naturally in Heaviside's scheme as functions of the internal or unimpressed parts of the E, H fields:

$$\nabla \times (H - h) = 4\pi J \quad (10)$$

$$-\nabla \times (E - e) = 4\pi G. \quad (11)$$

"We have now", Heaviside continued, "a dynamically complete system". [Heaviside, 1892 (1885b): 418] Or, rather, it will be complete when we have an equation expressing the energetic action of the impressed forces. "We must find", Heaviside remarked, "the rate of working of the impressed forces". This rate measures the energy fed into the system from without and therefore governs the development of the system over time.

I. The "Equation of Energy and Its Transfer"

a. The Role of Impressed Forces

Heaviside's equations easily yield the rate of working because the rate is simply the sum $\mathbf{e} \cdot \mathbf{J} + \mathbf{h} \cdot \mathbf{G}$ – the rates at which the impressed field forces $\mathbf{e}, \mathbf{h}$ feed energy into the 'currents' $\mathbf{J}, \mathbf{G}$. Taking, respectively, the scalar products of $\mathbf{e} - \mathbf{E}$, $\mathbf{h} - \mathbf{H}$ with (10), (11), adding and rearranging terms yields:

$$\mathbf{e} \cdot \mathbf{J} + \mathbf{h} \cdot \mathbf{G} = \mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{G} + \frac{[(\mathbf{H} - \mathbf{h}) \cdot \nabla \times (\mathbf{E} - \mathbf{e})] - [(\mathbf{E} - \mathbf{e}) \cdot \nabla \times (\mathbf{H} - \mathbf{h})]}{4\pi}. \quad (12)$$

But we already know that $\mathbf{E} \cdot \mathbf{J} + \mathbf{H} \cdot \mathbf{G}$ is just

$$Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t}.$$

Further, the term in curly brackets is identically equal to $\nabla \cdot \{(\mathbf{E} - \mathbf{e}) \times (\mathbf{H} - \mathbf{h})\}$. Whence, Heaviside concluded, the convergence of the vector

$$\mathbf{W} = \frac{(\mathbf{E} - \mathbf{e}) \times (\mathbf{H} - \mathbf{h})}{4\pi}$$

represents the rate at which energy flows out of a volume as a result of internal system processes. The equation of energy may then be written "in its most significant form" as:

$$\mathbf{e} \cdot \mathbf{J} + \mathbf{h} \cdot \mathbf{G} = Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{W}. \quad (13_H)$$

If, Heaviside noted, we integrate over all space then “ $\mathbf{W}$ goes out; or the total work per second of the impressed forces equals the total dissipativity plus the rate of increase of the total electric and magnetic energy”. [Heaviside, 1892 (1885b): 449]

Heaviside’s “vector rate of energy transfer”, or $\mathbf{W}$, and the Poynting vector are similar quantities. Poynting’s paper on energy flow in the field was read on Jan. 10, 1884, while this analysis – Heaviside’s first in print on the subject – is dated a year later, Feb. 21, 1885. Heaviside nevertheless maintained his deduction of the energy flow theorem to have been independent of Poynting’s. One can hardly doubt his claim in view of the fundamental role which energy considerations play in his overall reformulation of the Maxwell theory. Without an energy flow vector, or energy ‘current’ as he called it, Heaviside’s systematic reliance on energy would collapse since it was predicated on the assumption that the system transfers energy internally from point to point.

Heaviside’s explicit distinction between systemic and impressed (external) forces marks a major point of difference between him and Poynting. Poynting’s version of (13_H) (for matter at rest) can be put in the form: [Poynting, 1900 (1885)]

$$\frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} + \mathbf{E} \cdot \mathbf{C} + \frac{\nabla \cdot (\mathbf{E} \times \mathbf{H})}{4\pi} = 0. \quad (13_P)$$

Poynting’s (13_P) evidently lacks the term $\mathbf{e} \cdot \mathbf{J} + \mathbf{h} \cdot \mathbf{G}$, which is hardly surprising since Poynting made no distinction between systemic and impressed forces: his equations parallel to Heaviside’s (10)–(11) lack the vectors $\mathbf{e}, \mathbf{h}$.

Modern versions of equations (13) agree with Poynting’s (13_P). The sources of the difference between the two versions are the circuital laws (10)–(11). Without $\mathbf{e}, \mathbf{h}$ these laws effectively assume that all $\mathbf{E}$ fields are due to $\mathbf{G}$ currents and that all $\mathbf{H}$ fields are due to $\mathbf{J}$ currents. Heaviside wished to separate those sources of $\mathbf{E}, \mathbf{H}$ fields which do not (or may not) derive from $\mathbf{G}, \mathbf{J}$ currents from those which do. His goal

in proceeding this way was to be able to specify the sources which feed energy into the electromagnetic system independently of constraints imposed by the field equations. A voltaic source, for example, can be assumed constant independently of the processes which occur in the field, or the polarization of a permanent magnet may remain constant whatever else occurs. In Poynting's energy-balance equation (13_p) the right-hand side vanishes because all the field sources are currents (in Heaviside's sense); in Heaviside's (13_H) the right-hand side does not vanish because his circuital equations explicitly permit non-current sources. Nevertheless the true difference between equations (13) is not so much the presence of $\mathbf{e}, \mathbf{h}$ as their different estimations of the Poynting flux itself.

We can see this more clearly by removing the $\mathbf{e}, \mathbf{h}$ terms in (13_H) from $\mathbf{W}$ and transposing them to the left-hand side. Using the circuital relations (10)–(11) we find that (13_H) becomes:

$$\mathbf{H} \cdot (\nabla \times \mathbf{e}) - \mathbf{E} \cdot (\nabla \times \mathbf{h}) = Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} + \frac{\nabla \cdot (\mathbf{E} \times \mathbf{H})}{4\pi}. \quad (13'_H)$$

Equations (13'_H) and (13_H) are formally equivalent but they require different interpretations. According to Heaviside's equation (13_H) the Poynting flux does not involve impressed forces, which act to feed energy from without into the system. But according to (13'_H) the Poynting flux does involve the impressed forces; moreover, instead of treating these forces as feeding energy into the system, in (13'_H) we treat them as creating effective 'currents' $-\nabla \times \mathbf{e}, \nabla \times \mathbf{h}$ on which the field proper acts. That is, we assume that energy is abstracted from the system at the rate $\mathbf{H} \cdot [-(\nabla \times \mathbf{e})] + \mathbf{E} \cdot [\nabla \times \mathbf{h}]$ as a result of $\mathbf{E}, \mathbf{H}$ having to work upon $\nabla \times \mathbf{h}, -\nabla \times \mathbf{e}$. (Here we have an example showing how Heaviside's understanding of fields as 'working' on 'currents' focuses the analysis upon the proper form of the 'currents'.)

This alternative view of (13_H) would have made little physical sense to Heaviside since he required that $\mathbf{e}, \mathbf{h}$ – and so $\nabla \times \mathbf{e}, \nabla \times \mathbf{h}$ – be independently specifiable. One must be permitted to assume that they are constant, and this would be at best difficult to do if their curls were treated as currents which the fields can work upon. But only this latter interpretation permits defining the Poynting flux as $\nabla \cdot (\mathbf{E} \times \mathbf{H})/4\pi$. Re-

jecting it means defining the energy flux in Heaviside's manner as a cross-product from which the impressed forces $\mathbf{e}, \mathbf{h}$ have been removed. Heaviside's desire separately to represent energy fed into the field by external sources led directly to a different definition of the energy flux from Poynting's (and the modern) definition. Insofar as equations (13) are concerned either definition may be correct since both follow consistently from the field equations. Moreover, only in places where impressed force acts do the fluxes differ from one another.

b. The Energy Current

In his 1885 paper Heaviside introduced the concept of an "energy current" – a flow of energy through the field – well before he provided a mathematical foundation for it. He had first argued that a conducting wire must be seen as a "sink into which the energy is poured from the dielectric and there wasted, passing from the electrical system altogether". To make the idea more concrete, he envisioned a system consisting of a non-conducting dielectric studded with regions more or less conducting. In some of these conducting regions we have "impressed forces". In the steady state, he argued, a current of energy must flow through the dielectric from the loci of impressed force and nearly parallel to the surfaces of the conducting regions. But near conductors which are in circuit with a locus of impressed force the energy currents must have a slight slant towards the surfaces, ultimately meeting them perpendicularly by means of a sharp turn. So, for example, in the case of a current-bearing wire the energy flows normally in at the surface, but it is transmitted from the loci where it is born from the actions of impressed forces along paths nearly parallel to the wire – in this way we see how the energy proper is transmitted from its origins through the field.

Heaviside was obscure in details about the transient state, but his essential point is quite clear. In the initial moments the lines of energy current must themselves be in motion since "It takes some time to set the electric machinery going steadily", [Heaviside, 1892 (1885b): 438–41] and, therefore, time for the pattern of energy flow to stabilize. This represents electromagnetic induction. Consider, e.g., a conductor (in the steady state of the system) which is not in circuit with a lo-

cus of impressed force. The energy currents run parallel to its surface since it sinks no energy. But in the variable state the region does sink energy as a result of the electric currents generated within it by induction – whence in this state energy currents must converge upon it, so that their pattern differs from what it becomes in the steady state. Energy currents not only carry energy, they may themselves move.

Heaviside's image of the energy current rather sharply distinguishes his views from those of several of his Maxwellian contemporaries, and in particular from Poynting himself. Poynting, I have argued [Buchwald, 1985: Part. I, chap. 4], as well as J. J. Thomson afterwards, articulated a scheme in which tubes of induction, rather than energy currents, are reified. In such a theory a tube of induction in motion carries energy along with itself. Heaviside, by contrast, ignored induction tubes and concentrated instead upon tubes of energy flow. Since both approaches follow from equivalent sets of field equations, they are formally the same (at least up to the regions which act as energy sinks and within which material processes may occur which could distinguish the pictures). However, they differ substantially in their emphases. Heaviside's image directly concerns energy flow; Poynting's and J. J. Thomson's images concern energy flow only indirectly.

This difference starkly illuminates Heaviside's intellectual isolation from the Maxwellian mainstream. Unlike many of his contemporaries, he was not concerned with providing a method for understanding conductivity. Poynting's and Thomson's images of induction tubes were designed primarily to illustrate how one can envision conduction as the dissolution of induction and, consequently, as a process which dissipates energy. For Heaviside dissipation was a bare fact lying at the foundation of Maxwell's theory.

This intellectual isolation is hardly surprising. Heaviside not only rejected basic techniques employed by other Maxwellians, he was not deeply concerned with the problems the techniques were used for or with problems of understanding raised by the radical conception of conduction implicit in Maxwell's theory. Combine these characteristics with his dislike of controversy (though he was a formidable foe) and his early withdrawal to the parental home, and one should be surprised not at his isolation but at the extent of his contacts. He maintained a long and lively correspondence with FitzGerald and Lodge, and through them his reputation was furthered as they acted as inter-

mediaries between him and the larger Maxwellian community. A letter he wrote in 1888 to Lodge sadly shows how fearful Heaviside was of scientific authority despite his often sharply sarcastic remarks in print – indeed, the sharpness of the sarcasm was honed by his isolation. Rayleigh had criticized him for not sufficiently acknowledging others' work. Deeply troubled by this, Heaviside wrote Lodge:

As regards my setting the example of not acknowledging previous work, if this refers to something different from the (erroneously) supposed ignorance of M[axwell], I can honestly say I have not done so knowingly. I am probably a great sinner, but I do the best I can. It is a fact that I formerly omitted to publish solely on account of the priority difficulty. But more lately I have decided that I may as well publish, & take my chance. My consolation is that I am sinned against as well as sinning. Lord R's offer of "recognition" when he "writes again on the subject" is, considering the opinion he holds regarding me, very kind indeed. I suppose he would have great cause of complaint if it were my ignorance of his work that were examined into. But, on the other hand, I imagine that it can make no difference to one of Lord R's scientific standing whether he is properly acknowledged by an ignorant writer like myself or not, whilst it is far otherwise when it is the other way, a man at the top helping one at the bottom, or only a little way up the hill, & who, at the best, will never get more than $\frac{1}{4}$ of the way. So that Lord R. will place me under obligations, which it may never be in my power to return. [Heaviside to Lodge, UCL Lodge Collection]

This is hardly a calm and reasoned reaction to Rayleigh's criticism. It verges near the end on self-pity, to which Heaviside was occasionally prone. But his work also "bristles", as one reviewer put it [Swinburne, 1894], "with humour of a type he has invented". Not a true member of the Maxwellian mainstream, yet acknowledged by Maxwellians, Heaviside created for himself a unique niche. He occasionally suffered from his social and intellectual isolation, but he compensated with a brisk, biting humour, which acted as a guard against potential attack, thereby making it possible for him to publish even though he feared criticism.

V. *The Role of Subsidence*

Heaviside's account of Maxwell's theory appeared serially in the *Electrician* over a three-year period, comprising in the end forty-seven parts. In spring 1887 the series was in danger of being stopped, perhaps because of the editor's (C. H. W. Biggs) fears that Heaviside's occasionally strong language might be libelous – towards the series'

end Heaviside became involved in a nasty dispute with W. H. Preece concerning the role of self-induction in signal transmission. The series was eventually terminated at the end of March, 1888 by a new editor, N. H. Snell, who could find no one who ever read it.⁴

Despite these difficulties Heaviside's reformulation of Maxwell's theory was essentially complete by the end of the series. Its most novel features – the absence of the vector potential and the introduction of magnetic conductivity – reflected the theory's close integration with energy concepts. One sees this with particular clarity in Heaviside's account of the subsidence of electric displacement.

Displacement subsidence was a major feature of all Maxwellian theories, including – perhaps especially – Heaviside's. For this process explained 'charge' as a direct byproduct of field processes. In essence, 'charge' reflects – or rather, literally is – a discontinuity in the $\mathbf{D}$ (displacement) vector which arises in regions wherein, a continuous $\mathbf{D}$ field having been established initially, the field subsequently decays at different rates from point to point due to gradients in the ratio of conductivity to inductive capacity [Buchwald, 1985: Part I]. Both Poynting's and J. J. Thomson's employment of moving tubes of induction were in part attempts to make physical this otherwise obscure process by equating decay to the destruction of induction tubes [Buchwald, 1985: Part II]

Heaviside, unlike Poynting and Thomson, used energy considerations directly to analyse subsidence by starting with equation (13_H) of energy. [Heaviside, 1892 (1885b): 531–6] Assume that no impressed forces act on the system. Integrating over all space (13_H) becomes:

$$\int \left\{ Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} \right\} d^3x = 0.$$

Consider with Heaviside a case in which μ is zero, so that $\partial T/\partial t$ vanishes. Assume further that the operator $\partial/\partial t$ acting on a vector multiplies it by a constant factor n , e.g. $\partial \mathbf{E}/\partial t = n\mathbf{E}$. Then the global energy equation becomes:

$$\int \left\{ Q + \frac{\partial U}{\partial t} \right\} d^3x = \int \left\{ k + \frac{\epsilon n}{4\pi} \right\} E^2 d^3x = 0. \quad (14)$$

Consequently either E is everywhere zero, or else $k + \epsilon n/4\pi$ is somewhere positive, somewhere negative (or zero everywhere). Exclude the degenerate case and define p as $\epsilon/4\pi$. Let k_1/p_1 , k_2/p_2 be, respectively, the minimum and maximum values of the ratio k/p . Then it follows that $k_1/p_1 + n < 0$ while $k_2/p_2 + n > 0$ (since k and p are necessarily positive), so that $-k_1/p_1 > n > -k_2/p_2$, whence n is negative. Consequently the displacement D , equal to ϵE , decays as e^{nt} : subsidence emerges directly from Heaviside's "equation of energy". Heaviside developed the process in much greater detail than this short example suggests, but he always relied directly on the energy equation.

Subsidence, or field decay, also played a much greater practical role in Heaviside's work than in the work of other Maxwellians. In 1889 Heaviside developed a general solution to the field equations (in the absence of impressed forces) which incorporated magnetic conductivity and directly expressed field subsidence. To do so he employed an intricate operator notation which permitted expressions of the form e^{qt} where q is an operator function and t a variable. (This can be done by expanding e^{qt} in series, treating q as an algebraic variable rather than as an operator.) [Lützen, 1979 discusses aspects of Heaviside's operator calculus.] The field equations (10)–(11) become: [Heaviside, 1892 (1889): 468]

$$\nabla \times H = \left(4\pi k + \epsilon \frac{\partial}{\partial t} \right) E \quad (10')$$

$$-\nabla \times E = \left(4\pi g + \mu \frac{\partial}{\partial t} \right) H. \quad (11')$$

Define the quantities ρ, σ, ν and the operator q^2 as:

$$\rho = \frac{4\pi k}{2\epsilon} + \frac{4\pi g}{2\mu} \quad (15)$$

$$\sigma = \frac{4\pi k}{2\epsilon} - \frac{4\pi g}{2\mu} \quad (16)$$

$$\nu = \frac{1}{\sqrt{\mu\epsilon}} \quad (17)$$

$$q^2 = -[\nu \nabla \times]^2 + \sigma^2. \quad (18)$$

Then, Heaviside continued, if E_0, H_0 are the fields at time $t=0$, the fields at time t will be:

$$E = e^{-qt} \left[\left[\cosh(qt) - \frac{\sigma}{q} \sinh(qt) \right] E_0 + \left[\frac{\sinh(qt)}{q} \right] \frac{\nabla \times H_0}{\epsilon} \right] \quad (19)$$

$$H = e^{-qt} \left[\left[\cosh(qt) + \frac{\sigma}{q} \sinh(qt) \right] H_0 - \left[\frac{\sinh(qt)}{q} \right] \frac{\nabla \times E_0}{\mu} \right]. \quad (20)$$

Assume that the initial fields E_0, H_0 have no curl, or are 'polar'. Then q^2 reduces to σ^2 and (19)–(20) become:

$$E = E_0 e^{-\frac{8\pi kt}{2\epsilon}}$$

$$H = H_0 e^{-\frac{8\pi gt}{2\mu}}.$$

That is, with initially polar fields pure subsidence occurs, the E field subsiding because of electric conductivity, and the H field subsiding because of magnetic conductivity. Moreover, neither E nor H has curl, so that there are no currents G or J . This at once – and symmetrically – expresses the facts of decay and has an important practical significance for analysing signal distortion in wires.⁵

VI. *Electromagnetic Stress and Energy Flux United*

There are only two things going, Matter and Energy. Nothing else is a thing at all; all the rest are Moonshine, considered as Things. [Heaviside, 1892 (1886): 91]

Throughout his reformulation of Maxwell's theory Heaviside never made use of advanced dynamics. Nor did he even mention that Lagrange's equations or Hamilton's principle could be applied to electromagnetism. In part he was able to avoid advanced dynamics by, as

I remarked above, also avoiding situations where some such technique seemed to be necessary. But Heaviside also felt strongly that no situation required elaborate methods which, in his view, obscured the physics by imposing too complex a scheme upon the energy transformations. He therefore sought to create a method which could accommodate many of the same situations in electromagnetism as Lagrangian dynamics but which would not move too far from the essential physics of energy transformation.

One of the major questions for Maxwellians to answer concerned the stresses generated by the electromagnetic field, a question Maxwell had investigated by assuming the forces to be known rather than by starting with the energy densities of the field – though Maxwell was well aware that the stresses are implied by the behavior of the energy densities. Suppose, for example, one wanted to discover the stresses which act on a body in a magnetic or electric field. J. J. Thomson investigated precisely this question in 1888, [Thomson, 1968 (1888): secs. 30–9] and he relied directly upon Lagrangian techniques. His method, in essence, required expressing the complete energy density of the body – mechanical as well as electromagnetic – in terms of two sets of coordinates: one set represents the magnetic state of the field within the body, while the other set represents the strains within the body. To find the effect due to a change in any one of these coordinates Thomson simply applied Lagrange's equations to the coordinate in question. He was in this way able to investigate the strain of the body in a magnetic or electric field and to show that terms arise in addition to those required by Maxwell's stress, these terms being due to the dependence of inductive capacity and magnetic permeability upon strain. (Thomson did not, however, provide a new, more general expression for the stress than Maxwell's: he investigated only the modification to the strain, from which the modified stress can be found.)

Heaviside was well aware of Thomson's methods, and he of course disliked them intensely. In his opinion proceeding in this way one loses sight almost entirely of the physical fact that the effects which emerge from the elaborate Lagrangian analysis reflect energy transformations. Yet Thomson's was the only known method for investigating the stresses required by a given energy distribution. Since the stresses are direct physical reflections of the energy pattern Heaviside clearly had to provide an alternative method or else to confess that his

system was incapable of encompassing major classes of phenomena which hardly go beyond the usual boundaries of investigation (unlike, e.g., the Kerr, Hall and Faraday effects). He achieved this goal in 1891 when he not only gave a method for finding the stress but also linked the stress with the flux of energy represented by the Poynting vector. This major (and by modern standards spurious) unification relied directly upon his conception of impressed force and upon the assumption that a field pattern carried by moving matter cannot be altered solely as a result of uniform motion.

a. "Motional Forces"

The fields generated by motion first appeared in Heaviside's work with little discussion in 1885. [Heaviside, 1892 (1885b): 445–6 & 448] At that time he introduced what he later termed 'motional' electric (E_M) and magnetic (H_M) forces. If, he noted in 1885, motion with a velocity v occurs relative to a field of displacement (D) or of induction (B) then the following fields arise:

$$E_M = v \times B \quad (21)$$

$$H_M = D \times v. \quad (22)$$

Since Heaviside did not extensively discuss these 'fields' in 1885 one can interpret them in at least three not necessarily incompatible ways. The velocity v may refer to the ether proper, to matter, or to the fields D, B . In 1885 the velocity certainly referred at least to matter, since he applied (22) to a conductor moving through an electric field. [Heaviside, 1892 (1885b): 444 – "... motion of the circuit itself in a previously undisturbed electric field".] In that case H_M is the magnetic force induced "in a wire" by its motion. Similarly, motion of a wire through a magnetic field involves by (21) an "E.M.F. induced in a linear circuit". [Heaviside, 1892 (1885b): 448]

Heaviside initially justified (21)–(22) on the grounds that the E or H field around a circuit depends always on the changing B or D flux through it, however the change is produced. He did, however, distinguish without much discussion between two asymmetrical kinds of situations: one in which, e.g., a charged bullet moves along the axis of

an iron ring which is stationary with respect to the ether, the other in which the bullet is stationary in the ether while the ring moves. According to (22) an H_M field might be generated in either case, but the question is where do we locate the field – which effectively means what is v the velocity of. In 1885 Heaviside seems not to have thought through the problem since his remarks on the subject are somewhat ambiguous.

The ambiguity disappeared in 1891. He then carefully specified that the velocity in (21)–(22) must be the velocity of the ether proper. [Heaviside, 1950 (1891): sec. 44–56] He provided an illuminating example. Consider a charged body at rest relative to the ether, while the ether itself has a velocity v relative to an absolute reference frame. The uniform motion of the ether, Heaviside argued, should not in itself affect the pattern of displacement which diverges from the charge – that is, the charge must be unaltered by the motion:

That this should be so in a rational system we may conclude from the relativity of motion (the absolute motion of the universe being quite unknown, if not inconceivable), combined with our initial assumption that the electric flux (and the magnetic flux not here present) represent states of the medium which may be carried with it just as states of matter are carried with matter in its motion. [Heaviside, 1950 (1891): sec. 53]

This requirement is guaranteed by (22) with the proper understanding of the velocity. At any point P fixed absolutely there would be a displacement $\partial D/\partial t$ equal to $-(v \cdot \nabla)D(P)$. This involves, outside the charge proper, a magnetic field $v \times D$. But the magnetic force H_M due to motion of the ether in our example exactly annuls this, so that there is no net magnetic field anywhere when the charge moves with the medium. If, per contra, the medium is at rest and the charge moves through it with a velocity v , then everywhere there is a magnetic field equal to $v \times D$ because everywhere we have $\partial D/\partial t$ but no compensating motional field since the ether is at rest.

Heaviside's "motional fields" date from 1885. In 1890 Heinrich Hertz deduced a set of field equations for bodies in motion which incorporate Heaviside's motional fields together with other terms. Hertz had evidently not seen Heaviside's 1885 work,⁶ but his approach was nevertheless quite similar to Heaviside's. Ole Knudsen has recently discussed Hertz's work, [Knudsen, 1980] and he notes that Hertz's field equations for moving bodies were based on the assump-

tion that motion of the ether per se has no effect on the field pattern. In addition Hertz assumed that the velocity of the ether in a material body is the same as the velocity of the body itself. If we ignore this second assumption for a moment, then Hertz's equations are similar to Heaviside's. In Hertz's units we have:

$$-\mathbf{A}\nabla\times\mathbf{E} = \frac{\partial\mathbf{B}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{B}) + \mathbf{v}(\nabla\cdot\mathbf{B}) \quad (23)$$

$$\mathbf{A}\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} - \nabla\times(\mathbf{v}\times\mathbf{D}) + \mathbf{v}(\nabla\cdot\mathbf{D}) + 4\pi\mathbf{C}. \quad (24)$$

In (23)–(24) $\mathbf{v}$ is the ether's velocity. These equations incorporate Heaviside's 'motional' fields (21)–(22) and have in addition convective sources $\mathbf{v}(\nabla\cdot\mathbf{B})$, $\mathbf{v}(\nabla\cdot\mathbf{D})$. Since, for Hertz, $\mathbf{v}$ is also the velocity of a material body (the ether is fully dragged), the convection current $\mathbf{v}(\nabla\cdot\mathbf{D})$ always involves motion of the ether.

This interpretation differs from Heaviside's. For him the term $\mathbf{v}(\nabla\cdot\mathbf{D})$ referred to the velocity of "moving electrification", and this is identical to the velocity of the ether only "when the medium and the charge move together". [Heaviside, 1950 (1891): 14] This implies that matter need not fully drag ether since $\nabla\cdot\mathbf{D}$ occurs only where there is inhomogeneity and conductivity, and conductivity requires matter. For Heaviside the convection current derived from the motion of 'electrification' in the Maxwellian sense of the term, and this need not involve motion of the ether proper but only of matter.

b. The Electromagnetic "Equation of Activity"

Heaviside utilised the 'motional' fields to construct a general equation for energy which yielded simultaneously the electromagnetic stress tensor and the Poynting energy flux. He naturally regarded this result as the capstone to his system, but it was almost universally ignored, primarily because his presentation was, even by his standards, wonderfully obscure. Rayleigh, as editor of the *Philosophical Transactions*, wrote Heaviside:

Both our referees, while reporting favourably upon what they could understand, complain of the exceeding stiffness of your paper. One says it is the most difficult he ever

tried to read. Do you think you could do anything viz. illustrations or further explanations to meet this? As it is, I should fear that no one would take advantage of your work. [Appleyard, 1929: 227–8]

Horace Lamb, in a review of Heaviside's *Electrical Papers*, remarked:

The last paper (but one) in the book forms a sort of crown to the whole ... Unfortunately this paper is by far the hardest to read. Free use is made of the scalar and vector products of Hamilton, but the author is careful to give us his emphatic opinion that quaternions proper are unsuited to the purposes of mathematical physics. This courageous declaration will, we fear, cause a wicked joy in the hearts of many who have struggled in vain with the refractory symbols. For the special system of mathematical shorthand affected by Mr. Heaviside there is much to be said, but for our own part we should prefer to have papers which profess to give new and important results written in the more homely language of "Mr. Cartesian". Another prominent feature in this memoir is the frequent appeal to the principle of "continuity of energy", but this imposing phrase seems to mean nothing more nor less than Maxwell's negation of action at a distance ... [Lamb, 1893: 505]

Obviously Heaviside's 'crowning' work was extremely difficult for his contemporaries, in part because he used complicated and unfamiliar vector manipulations throughout. But there is a deeper source of the problem, hinted at in the last sentence of the quotation from Lamb's review. What Heaviside conceived of as a full-fledged method which unified stress and energy flux, Lamb saw only as a "negation of action at a distance". Here, then, we have found a major point at which Heaviside diverged markedly from his Maxwellian contemporaries.

Heaviside's analysis is, by any standards, extraordinarily difficult. To understand it we shall begin in the middle, with his deduction of a general electromagnetic "equation of activity" or equation of energy balance. Having introduced a new system of units such that in equations (1)–(12) every factor 4π is replaced by unity (and 8π by 2), he wrote the field equations as follows:

$$\nabla \times (H - h_0 - h) = J \quad (25)$$

$$-\nabla \times (E - e_0 - e) = G, \quad (26)$$

wherein

$$J = C + \frac{\partial D}{\partial t} + vq$$

$$G = K + \frac{\partial B}{\partial t} + v\sigma$$

h_0 = impressed magnetic force⁷

e_0 = impressed electric force

$h = D \times v$ (motional magnetic force)

$e = v \times B$ (motional electric force)

$C = kE$ (electric conduction current)

$K = gH$ (magnetic conduction current)

$\sigma = \nabla \cdot B$ (magnetification)

$p = \nabla \cdot D$ (electrification).

Heaviside's next step was essentially analytical. He permitted ϵ, μ ($D = \epsilon E$, $B = \mu H$) to depend upon time and position, thereby obtaining (27) from (25)–(26) – or, rather, from their extensions (28) and (29), given below – after considerable simplification:

$$\begin{aligned} e_0 \cdot J + h_0 \cdot G = & \left(Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} \right) + v \cdot (\rho E + \sigma H) + \frac{1}{2} E \cdot \left[\frac{\partial \epsilon}{\partial t} E \right] \\ & + \frac{1}{2} H \cdot \left[\frac{\partial \mu}{\partial t} H \right] - (e \cdot J + h \cdot G) + \nabla \cdot [E_1 \times H_1] \end{aligned} \quad (27)$$

where Q, U, T are the same quantities as in equations (2'), (4), (7), and E_1, H_1 are:

$$E_1 = E - e_0 - e \text{ hence } -\nabla \times E_1 = G \quad (28)$$

$$H_1 = H - h_0 - h \text{ hence } \nabla \times H_1 = J. \quad (29)$$

Heaviside's goal of a general 'activity' equation requires, for reasons which will soon become apparent, uncovering every term in (27) that contains a velocity even implicitly. This led him to adopt (though he did not here explicitly discuss the point) Hertz's assumption that v represents always the ether's velocity, which is the same as the velocity of a material body when the latter moves. So Heaviside could then write:

$$\frac{\partial \epsilon}{\partial t} = \frac{d\epsilon}{dt} - (v \cdot \nabla) \epsilon \quad (30)$$

$$\frac{\partial \mu}{\partial t} = \frac{d\mu}{dt} - (\mathbf{v} \cdot \nabla) \mu \quad (31)$$

wherein $\mathbf{v}$ is both the velocity of matter with capacity ϵ and permeability μ and also the velocity of ether within the matter.⁸

Further manipulation of (27) using (30)–(31) yields:

$$\begin{aligned} e_0 \cdot \mathbf{J} + h_0 \cdot \mathbf{G} = & \left(Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} \right) \\ & + \mathbf{v} \cdot \{ \rho \mathbf{E} + \sigma \mathbf{H} - \tfrac{1}{2} E^2 (\nabla) \epsilon - \tfrac{1}{2} H^2 (\nabla) \mu + \mathbf{J} \times \mathbf{B} + \mathbf{D} \times \mathbf{G} \} \\ & + \nabla \cdot \{ \mathbf{E}_1 \times \mathbf{H}_1 \} + \tfrac{1}{2} E^2 \frac{d\epsilon}{dt} + \tfrac{1}{2} H^2 \frac{d\mu}{dt} \end{aligned} \quad (32)$$

wherein $\mathbf{v} \cdot \{ (\nabla) \epsilon \}$ signifies $(\mathbf{v} \cdot \nabla) \epsilon$ and similarly for μ .

Heaviside did not in fact give equation (32). Instead, he introduced two auxiliary quantities, U_c and T_M , defined by the equations:

$$\frac{\partial U}{\partial t} c = \tfrac{1}{2} E^2 \frac{\partial \epsilon}{\partial t} \quad \text{and} \quad \frac{\partial T}{\partial t} M = \tfrac{1}{2} H^2 \frac{\partial \mu}{\partial t}.$$

Note that U_c and T_M are given only indirectly by their partial derivatives. In Heaviside's view these functions, defined only through their partial derivatives, represented the parts of the energy function which depend on variation with time in ϵ, μ . He wished to emphasize energy formulations wherever possible, so he chose what at first seems to be an innocuous short-hand. However, we shall shortly see that his basic goal of an electromagnetic activity equation which can be interpreted dynamically did not require either U_c or T_M . One can retain the form of equation (32) and still reach all of his conclusions.

I emphasize what might seem to be an unimportant point because Heaviside's functions U_c and T_M completely lack physical significance. Philip Lervig has demonstrated that they cannot be functions of the local variables and their (local) derivatives and yet satisfy Heaviside's requirements.⁹

If Heaviside's theory depended critically upon their existence then it would be mathematically faulty. Fortunately the theory does not require them. Unfortunately Heaviside did not give them sufficient thought, preferring to insert what looked like energy expressions without considering whether they could represent anything at all. His unusual sloppiness here reflects the constant impulsion he felt to recur to energy formulations wherever possible.

Even the lengthy equation (32) does not fully indicate every term in which the velocity may occur. To bring out the remaining terms, Heaviside defined four new vectors:

$$J_0 = J + \nabla \times h \quad (33)$$

$$G_0 = G - \nabla \times e \quad (34)$$

$$f = E\rho + H\sigma - \frac{1}{2}E^2\nabla\varepsilon - \frac{1}{2}H^2\nabla\mu \\ + (J + \nabla \times h_0) \times B - (G - \nabla \times e_0) \times D \quad (35)$$

$$F = E\rho + H\sigma - \frac{1}{2}E^2\nabla\varepsilon - \frac{1}{2}H^2\nabla\mu \\ + (\nabla \times H) \times B - (\nabla \times E) \times D. \quad (36)$$

These definitions permit (32) to be expressed in two analytically equivalent ways:

$$e_0 \cdot J_0 + h_0 \cdot G_0 = \quad (37)$$

$$\left(Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} \right) + v \cdot f + \nabla \cdot \{ E_1 \times H_1 - e \times h_0 - e_0 \times h \} \\ + \frac{1}{2}E^2 \frac{d\varepsilon}{dt} + \frac{1}{2}H^2 \frac{d\mu}{dt}$$

$$e_0 \cdot J_0 + h_0 \cdot G_0 = \quad (38)$$

$$\left(Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} \right) + v \cdot F + \nabla \cdot \{ E_1 \times H_1 - e \times (h + h_0) - e_0 \times h \} \\ + \frac{1}{2}E^2 \frac{d\varepsilon}{dt} + \frac{1}{2}H^2 \frac{d\mu}{dt}.$$

One of (37)–(38) is the correct electromagnetic equation of activity according to Heaviside and contains both the stress tensor and the Poynting flux.

c. The Dynamical Equation of Activity

Heaviside was at his most cryptic and difficult when he interpreted (37)–(38). Again we must compress to a great extent his intricate and obscure discussion. We begin with a general mechanical “equation of activity” for a system acted on solely by internal stresses. In such a system a volume receives energy in two ways: by convection from matter moving into the volume and by the action of stress on the volume’s surface; represent the volume density of the flow rates by $-\nabla \cdot \mathbf{W}$. Then if Q is the rate of dissipation within the volume, U is the internal potential energy, and T is the internal kinetic energy, Heaviside’s mechanical “equation of activity” is: [Heaviside, 1892 (1891): 537]

$$Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{W}. \quad (39)$$

Heaviside expressed the stresses in $\mathbf{W}$ explicitly by introducing, in effect, a stress tensor Q_{ij} (which he allowed to be anti-symmetric, thereby permitting a possible hidden angular momentum) such that the stress $\mathbf{P}_N$ on a plane with unit normal $\mathbf{e}_n$ is $\sum_i \mathbf{e}_i [\mathbf{e}_n \cdot \sum_j Q_{ij} \mathbf{e}_j]$ (wherein the $\mathbf{e}_i, \mathbf{e}_j$ are unit vectors along the coordinate axes). (That is, the stresses $\mathbf{P}_i$ on the coordinate planes whose normals are $\mathbf{e}_i$ are $\sum_j Q_{ji} \mathbf{e}_j$.) Whence the force $\mathbf{Y}$ per unit volume is:

$$\mathbf{Y} = \sum_i \mathbf{e}_i (\nabla \cdot \mathbf{Q}_i) \quad \text{wherein} \quad \mathbf{Q}_i = \sum_j Q_{ij} \mathbf{e}_j. \quad (40)$$

The rate at which the stress in the direction $\mathbf{e}_n$ transfers energy to the surface it acts upon is $-\mathbf{P}_N \cdot \mathbf{v}$. (The minus sign reflects Heaviside’s convention that a positive $\mathbf{P}_N$ is a pull, so that the energy transfer due to the stress is opposed to the motion.) Whence we have:

$$\text{energy transfer rate along } e_n = -e_n \cdot \sum_i v_i Q_i. \quad (41)$$

Or, in Heaviside's terms, $-\sum_i v_i Q_i$ is the 'activity' of the stress.

Given these results, Heaviside reformulated (39) in a way particularly significant for electromagnetism. First, we may at once write W as the sum of convected energy and energy due to the activity of the stress:

$$W = v(U+T) - \sum_i v_i Q_i. \quad (42)$$

We next use (40) to introduce the volume force (or 'translational' force as Heaviside called it) Y into (39) via (42) (since $\nabla \cdot \sum_i v_i Q_i = v \cdot \sum_i e_i \{ \nabla \cdot Q_i \} + \sum_i \{ Q_i \cdot \nabla \} v_i$):

$$Y \cdot v + \sum_i (Q_i \cdot \nabla) v_i = Q + \frac{\partial U}{\partial t} + \frac{\partial T}{\partial t} + \nabla \cdot \{ v(U+T) \}. \quad (43)$$

Heaviside further demonstrated that the term in (43) in v_i depends solely upon distortion and rotation (or upon distortion alone if Q_{ij} is symmetric).¹⁰ Whence (43) permits us to distinguish the activities due to distortion and rotation on the one hand, from the purely translational activity $Y \cdot v$, on the other. To do so Heaviside defined Q_0, U_0, T_0 as those parts of the dissipation and internal energies which alter solely as a result of translational action. We use a subscript one to denote the effect of distortion, and we represent the energy rate involved in rotation by the product $\tau \cdot \omega$ where τ is torque and ω is vorticity ($\frac{1}{2} \nabla \times v$):

$$Y \cdot v = Q_0 + \frac{\partial U_0}{\partial t} + \frac{\partial T_0}{\partial t} + \partial \cdot \{ v(U_0+T_0) \} \quad (44)$$

$$\sum_i (Q_i \cdot \nabla) v_i = Q_1 + \frac{\partial U_1}{\partial t} + \frac{\partial T_1}{\partial t} + \nabla \cdot \{ v(u_1+T_1) \} + \tau \cdot \omega. \quad (45)$$

We next imagine a particular system which has one unusual character-

istic: energy may be transmitted through it even though the velocity $\mathbf{v}$ vanishes. Heaviside did not discuss this point. However it would be entirely consistent with his general approach to assume that $\mathbf{v}$ represents only a gross or large-scale motion, while there may also be small-scale motions which themselves involve energy transfer. Represent the vector rate of transfer due to these small-scale effects by $\mathbf{S}$. Suppose further that external sources feed energy into the system at the rate R per unit volume. Then the equation of activity for this system will be:

$$R = Q_0 + \frac{\partial U_0}{\partial t} + \frac{\partial T_0}{\partial t} + \nabla \cdot \left[\mathbf{v}(U_0 + T_0) - \sum_i v_i \mathbf{Q}_i \right] + \quad (46)$$

$$Q_1 + \frac{\partial U_1}{\partial t} + \frac{\partial T_1}{\partial t} + \nabla \cdot \{ \mathbf{S} + \mathbf{v}(U_1 + T_1) \}.$$

Using (44) we transform (46) into (47):

$$R = Q_1 + \frac{\partial U_1}{\partial t} + \frac{\partial T_1}{\partial t} + \mathbf{Y} \cdot \mathbf{v} + \boldsymbol{\tau} \cdot \boldsymbol{\omega} + \nabla \cdot \left[\mathbf{S} + \mathbf{v}(U_1 + T_1) - \sum_i v_i \mathbf{Q}_i \right]. \quad (47)$$

Before examining how Heaviside applied (47) to electromagnetism, pause for a moment to consider its meaning. We envision a system whose structure is unknown to us but within which stress can act to produce motion. We separate the energy terms which involve distortion and rotation from those which involve pure translation, and we incorporate the latter into the expression $\mathbf{Y} \cdot \mathbf{v}$. Further, we assume that this system is also capable of transmitting energy without large-scale bodily motion; we represent this transmission by the vector $\mathbf{S}$. Or, to be more precise, the motion involved in $\mathbf{v}$ must be only a net motion; there may be other motions superposed on $\mathbf{v}$ which also transfer energy. Finally, R represents the insertion into the system of energy from external sources.

d. Electromagnetism and Dynamics

Heaviside now had to link the electromagnetic equations (37)–(38) with the dynamical equation (47). The first point to remark in the comparison is that the electromagnetic Q , $\partial U/\partial t$, $\partial T/\partial t$ in (37)–(38) evidently correspond to the Q_1 , $\partial U_1/\partial t$, $\partial T_1/\partial t$ of (47). That is, the usual electromagnetic energy expressions apparently represent distortional effects (though we know nothing of the actual systemic strains that are involved). The next point is the critical one: we have two possible equations for the vectors S and Y in (47):

$$S_1 + v(U_1 + T_1) - \sum_i v_i Q_i = E_1 \times H_1 - e \times h_0 - e_0 \times h \quad (37')$$

$$S_2 + v(U_1 + T_1) - \sum_i v_i Q_i = E_1 \times H_2 - e \times h_0 - e_0 \times h - e \times h \quad (38')$$

The final term in the electromagnetic equation of activity (32) – $\frac{1}{2}E^2 d\epsilon/dt + \frac{1}{2}H^2 d\mu/dt$ – must represent the effects of rotation. Though Heaviside used the unphysical quantities U_c, T_M to write the term as $d(U_c + T_M)/dt$, the important characteristic was not the presence of these quantities but the use of total or material derivatives rather than partial derivatives: the total derivative permitted Heaviside, given a certain restriction, to link the terms to vorticity ('spin' in his language) and torque when the stress satisfies equation (50) below.¹¹

Let's examine how the scheme works. Suppose we choose (38'). We must next divide the term between S_2 and $\sum_i v_i Q_i$. This may be done arbitrarily, but the division is subject to the condition that the resulting Q_i yield F . Expanding E_1, H_1 (equations (28)–(29)) we may, e.g., assert the following relations (also using the definitions of e, h following equation (26)):

$$S_2 = (E - e_0) \times (H - h_0) \quad (48)$$

$$\sum_i v_i Q_i = (v \times B) \times H + E \times (D \times v) + v(U_1 + T_1). \quad (49)$$

From (49) Heaviside could find Q_i and, from this, the force F via (40). The result for the force is, indeed, the expression given by (36). That

is, the division (48)–(49) yields a stress which is correctly related to the previously given translational force. In this case the stress P_N on a plane with normal e_n is:

$$P_N = \{H(B \cdot e_n) - e_n T_1\} + \{E(D \cdot e_n) - e_n U_1\}. \quad (50)$$

This is precisely the same as Maxwell's stress if the medium is isotropic. (That is, if the inductions are parallel to the forces.) Moreover, any other division of (38') between the energy flux vector and the stress fails since only this division yields a stress with the correct translational force.

Of course we could also have chosen (37') and examined possible expressions compatible with it. However, no division among the terms can yield a stress compatible with the translational force f . Whence we conclude with Heaviside that the flux S due to the hidden, small scale motions is necessarily given by (48), while the stress is required to be (50), and the translational force is (36), which we may rewrite as:

$$F = EQ + H\sigma - \frac{1}{2}E^2 \nabla \epsilon - \frac{1}{2}H^2 \nabla \mu + C \times B + \frac{\partial D \times B}{\partial t}. \quad (51)$$

Heaviside remarked of the term in (51) containing the cross-product of the displacement with the magnetic induction that it "is ... the momentum (translational, not magnetic, which is quite a different thing), provided the force F is the complete force from all causes acting ...". [Heaviside, 1892 (1891): 558] But, he continued, we have no knowledge whatsoever that F is the sole force acting upon the ether proper. There may be internal forces due to constraints unknown to us which have no effects upon the energies we are able to make use of.

Despite this indeterminacy, which has no practical effect, the stress proper correlated with F is completely determinate using Heaviside's method. This is not the case if we were, as he put it, to proceed (as Maxwell did) by examining solely the forces in statical circumstances. The introduction of the velocity v settles matters and is therefore a critical innovation. Heaviside remarked:

... there is really no decisive settlement possible from the theoretical statical standpoint, and nothing short of actual experimental determination of the strains produced and their

exhaustive analysis would be sufficient to determine the proper stress-function. But when the subject is attacked from the dynamical standpoint, the indeterminateness disappears. From the two circuital laws of variable states of electric and magnetic force in a moving medium, combined with certain distributions of stored energy, we are led to just one stress-vector. [Heaviside, 1892 (1891): 569]

It is hardly surprising that Heaviside's Maxwellian colleagues found this work hard going. In order to grasp it three things at least are essential: first, patience with the novel vector symbolism Heaviside used; second, an understanding that Heaviside's goal was to avoid Lagrangian methods; and, third, extreme patience in deciphering Heaviside's highly compressed discussion. Few Maxwellians were sympathetic to vector notation, and even fewer knew Heaviside's goal or, if they did know it, neither sympathised with it nor had the patience to follow the crooked path he charted. This divergence between Heaviside and the Maxwellians reflects in part a significant difference in their conceptions of physical theory.

Compare the following two remarks. The first is by J. J. Thomson:

In general the way that we are able to connect various physical phenomena is by seeing from the behaviour of the system under certain circumstances that there must be a term of a certain kind in [the Lagrangian]; the existence of this term will then often by the application of Lagrangian or Hamiltonian methods point to other phenomena besides the one that led to its detection. [Thomson, 1968 (1888): sec. 6]

The second is by Heaviside:

The important thing to be grasped is, that whenever we definitely localize energy we can obtain an equation showing its continuity in space and time, and that when we can only partially localize it, we can still, by proper devices, allow for the absence of definiteness. The equation is simply the equation of activity of the dynamical system, suitably arranged to show the flux of energy, and can always be obtained when the equations of motion are known and also the nature of the stored and wasted energies ... in the electromagnetic case the "equations of motion" are the two circuital laws ... [Heaviside, 1950 (1891): secs. 69–70]

Thomson and Heaviside were advocating distinctly different methods of theorizing. With Thomson's method we begin directly with energy densities and then all else follows by cranking the Lagrangian or Hamiltonian machinery. With Heaviside's method we must independently possess what he terms the system's "equations of motion" (viz. whatever equations experience gives us to link our variables) as well

as the energy densities; then we manipulate these equations and densities until we obtain an “equation of activity” which can be interpreted dynamically as expressing a stress and an energy flux. Put perhaps too bluntly, Thomson proposed to deduce everything, including Heaviside’s equations of motion, from advanced dynamics. Heaviside proposed to acquire the equations of motion from experience and then to investigate the stresses and energy fluxes by manipulating the equations into a form that can be interpreted dynamically.

Thomson’s method is more powerful than Heaviside’s because it can be used to discover new effects – in fact new equations of motion – attendant on modifications to the energy densities. Heaviside’s method requires that these equations always be given first. But where Thomson’s method abstracts almost completely from the energy transformations which occur, Heaviside’s method exhibits them quite directly because his “equation of activity” is an expression of energy transformation. Thomson’s use of Lagrangian techniques, though more extensive than was common among Maxwellians, was otherwise quite usual.¹² Indeed, it was the only method available until the mid-1890’s for attacking problems at the frontier of research in electromagnetism. Heaviside cared little about these kinds of investigations, so a major attraction for others of the Lagrangian technique had no effect upon him.

VII. Epilog

My dear Heaviside

I am very sorry to hear that you are left alone. I do not seem to feel anything “unaccountable” in your being sorry. Man was not meant to live alone ... I have a brother, a clergyman, who is unmarried and lives alone and who keeps 3 cats. They are the plague of his life. They mew to get in when he is reading and to get out when he is dozing and they misbehave themselves and make stinks, a thing he abhors, in his sitting room, and yet this is just what fills a place in his nature and makes them indispensable. It is a law of our nature that if we are not taking trouble for others we will become beasts, and so we are compelled to value those who fill this void and save us from ourselves. [FitzGerald to Heaviside, Dec. 7, 1896, {IEE Collection} written shortly after the death of Heaviside’s father.]

Heaviside’s *Electrical Papers* appeared in 1892, and the three volumes of his *Electromagnetic Theory* in 1893, 1899 and 1912. But his work on

Maxwell's theory was essentially complete by 1891, and he was never able fully to grasp the many changes which rapidly occurred in electromagnetism after 1895. [Buchwald, 1985: Epilog] In fact Heaviside's published work after 1896, the year his father died, shows little fundamental change. He was deeply affected by the loss of his parents (his mother died in 1894), who had shielded him from the necessities of life, as one can see from FitzGerald's moving letter of consolidation. He did thereafter keep up a lively correspondence, particularly with FitzGerald (who was his main contact with the wider Maxwellian community), but FitzGerald himself died young in 1900. After that Heaviside's meager contact with ongoing research rapidly evaporated altogether. But he had always been interested for the most part only in problems concerning the structure of Maxwell's theory as he understood it. That theory, he was well aware, quickly altered beyond recognition after c. 1895. He never entered with any real sympathy into developments in physics after that.¹³

Heaviside was perhaps the last auto-didact to have a significant impact on the development of physics. He was eccentric to an almost absurd degree, but British Victorians, despite their present reputation for stodginess, were often more willing than censorious moderns to tolerate an original but unconventional mind. How well would a man fare among physicists today who never went to meetings or visited colleagues, refused grants, was argumentative, irascible, attached to novel symbolisms which almost no one could follow, and who was almost completely uninterested in the latest discoveries?

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NOTES

1. Though from 1896 on he held a Civil List Pension of 120 pounds yearly "in consideration of his work in connexion with the theory of electricity", which was increased by 100 pounds in 1914: Searle, 1950: 95.
2. Heaviside almost certainly read Thomson's *Papers* (1872). See Wise, 1979 and 1981 for a discussion of Thomson's early role in applying energy concepts to electrodynamics and other areas.
3. I have discussed this aspect of Maxwell's theory, among others, in several recent articles. (Buchwald, 1979, 1981), but a more extensive discussion is in Buchwald, 1985.
4. Jordan, 1982a analyzes this controversy in detail and provides much insight into the resistance which mathematical theory encountered among 'practical' electricians in the mid 1880s. See especially pp. 450-6 for Heaviside's role in the debate.
5. If σ vanishes with initially polar fields and the ratios k/ϵ , g/μ are equal, then the solution (19)-(20) is precisely the same as the solution for no conductivity of any kind if, for the fields at time t in the latter case, we substitute these same fields multiplied by $e^{-\sigma t}$. The fields therefore subside, but without disturbing the non-conducting field pattern. See Heaviside, 1892 (1886): 135 and 1892 (1889): 470. Of course, g seems not to exist. However, Heaviside demonstrated (Heaviside, 1892 (1886): 136) that a pair of parallel conducting wires separated by a slightly conducting dielectric are governed by the same form of equations as the plane wave propagating in a magnetically and electrically conducting dielectric, provided we replace the ratio k/ϵ with R/L and the ratio g/μ with K/S , where R , K , L and S are, respectively, the wire's resistance per unit length, the dielectric's conductance per unit length, and the wire circuit's inductance and permittance per unit length. There will then be no signal distortion if R/L equals K/S - the conductance, K , of the dielectric replaces the magnetic conductivity, g . Heaviside had worked out the theory of the distortionless line without using g in 1887 (Heaviside, 1892 (1886): 119 & ff.), but g links the problem to the general structure of the symmetric field equations - thereby permitting use of the general solution incorporating g in practical situations simply by altering, in Heaviside's words, the "meaning of the symbols".
6. Hertz, 1962 (1890a): 196. Referring to Heaviside's 1888 paper he remarked: "These equations will be found in the *Phil. Mag.* for February 1888. Reference is there made to earlier papers in the *Electrician* for 1885, but this source was not accessible to me". The motional fields are not used in Heaviside's 1888 paper.
7. Heaviside used the word 'intrinsic' instead of 'impressed' here, but the context indicates that they mean the same thing - these forces are not due to internal field processes but are 'intrinsic' to matter.
8. Heaviside made the point explicit in the *Electromagnetic Theory* (Heaviside, 1950 (1891): sec. 71):

... electrification, which is always found associated with matter, moves with the medium, which is then the matter and ether, moving together.

However, Heaviside was unsure whether this requirement of a fully dragged ether could withstand optical evidence, with which he was not well acquainted. He remarked (Heaviside, 1950 (1891): sec. 71):

Whether, when matter moves, it carries the immediately surrounding ether with it, or moves through the ether, or only partially carries it forward, and what is the nature of the motion produced in the ether by moving matter, are questions which cannot be answered at present. Optical evidence is difficult of interpretation, and conclusions therefrom are conflicting.

9. Lervig's argument is quite simple. We assume that U_c , e.g., must be local in its variables, which in its case are E^2 , ϵ , $\partial\epsilon/\partial t$, ∇E^2 , and other similar functions of E and ϵ . This allows us to expand U_c in a Taylor's series, whose time derivative we can then compute

$$\begin{aligned} U_c &= \left[U_c^{(0)} + \left[\frac{\partial U_c}{\partial E^2} \right]^{(0)} E^2 + \left[\frac{\partial U_c}{\partial \epsilon} \right]^{(0)} \epsilon + \dots \right] \\ &+ \frac{1}{2} \left[\left[\frac{\partial^2 U_c}{\partial (E^2)^2} \right]^{(0)} (E^2)^2 + \left[\frac{\partial^2 U_c}{\partial \epsilon^2} \right]^{(0)} \epsilon^2 + 2 \left[\frac{\partial^2 U_c}{\partial E^2 \partial \epsilon} \right]^{(0)} \epsilon E^2 + \dots \right] \\ \frac{\partial U_c}{\partial t} &= \frac{\partial E^2}{\partial t} \left[\left[\frac{\partial U_c}{\partial E^2} \right]^{(0)} + \left[\frac{\partial^2 U_c}{\partial (E^2)^2} \right]^{(0)} E^2 + \left[\frac{\partial^2 U_c}{\partial E^2 \partial \epsilon} \right]^{(0)} \epsilon + \dots \right] \\ &+ \frac{\partial \epsilon}{\partial t} \left[\left[\frac{\partial U_c}{\partial \epsilon} \right]^{(0)} + \left[\frac{\partial^2 U_c}{\partial \epsilon^2} \right]^{(0)} \epsilon + E^2 \left[\frac{\partial^2 U_c}{\partial E^2 \partial \epsilon} \right]^{(0)} + \dots \right] + \dots \end{aligned}$$

Heaviside requires that $\partial U_c / \partial t$ must be equal to $\frac{1}{2} E^2 \partial \epsilon / \partial t$. Substituting this in our Taylor's series for $\partial U_c / \partial t$ and assuming that we can vary all the local variables independently yields impossible requirements for the cross-derivatives:

$$\frac{\partial^2 U_c}{\partial (E^2)^2 \partial \epsilon} = \frac{1}{2} \quad \text{and} \quad \frac{\partial^2 U_c}{\partial (E^2)^2 \partial \epsilon} = 0.$$

These requirements are obviously inconsistent.

The problem arises entirely from Heaviside's condition on the time derivative of U_c (and T_M). If we can reconstruct his equations without using these time derivatives, and still retain their significance, then we will have avoided the difficulty entirely. We can indeed do so, as I show below. Moreover, if the problem had been pointed out to Heaviside quite likely he would simply have done the same thing – removed U_c , T_M entirely since they are without immediate importance for his theory.

10. When Heaviside wrote of 'distortion', 'rotation' and 'translation' in this context he was not referring to the ultimate structure of the ether. Rather, these motions refer only to the velocity $\mathbf{v}$ which represents – see below – a large-scale motion on which the internal motions of the ether are superposed.
11. The term $\boldsymbol{\tau} \cdot \boldsymbol{\omega}$ reflects the existence of anisotropy in the inductive capacity or magnetic permeability (Heaviside, 1892 (1891): 549–50). If the stress tensor Q_{ij} is symmetric then this term vanishes. If it is not symmetric then the torque can be obtained from the equation:

$$\tau \times e_n = \sum_i e_i \left\{ e_n \cdot \sum_j Q_j e_j \right\} - \sum_i e_i \left\{ e_n \cdot \sum_j Q_j e_j \right\}.$$

Heaviside was, as usual, quite obscure about the way in which $\tau \cdot \omega$ emerges from the electromagnetic equation of activity. However comments he later included in his *Electromagnetic Theory* [Heaviside 1950 (1893): secs. 123–4] clarify his ideas. We must first limit our considerations to situations in which the system of displacement or magnetic induction moves as a rigid body. “This will occur”, Heaviside remarked [Heaviside (1950) (1893): sec. 124], “when the sources of the displacement (electrification, for instance) move as a rigid body, and the motion is of a steady type, so long continued that the displacement itself (in the moving system) is steady”. Under these circumstances we can calculate the material rate of change of the displacement (or magnetic induction), dD/dt , by considering only the rotation of the matter carrying D . That, one can show by standard arguments, is just $\omega \times D$, this then being “... the rate of time-variation of the displacement in a moving element of the displacement system, caused by the rotation”. Or, to generalize in Heaviside’s manner, we may replace the operator d/dt with $\omega \times$. From this we can calculate the value of $E \cdot dE/dt$:

$$E \cdot \frac{dE}{dt} = E \cdot \frac{dD}{dt} - D \cdot \frac{dE}{dt} = E \cdot (\omega \times D) - D \cdot (\omega \times E) = 2\omega \cdot (D \times E).$$

We may similarly find:

$$H \cdot \frac{dH}{dt} = 2\omega \cdot (B \times H).$$

Consequently the extra terms in the energy equation evidently represent a torque τ equal to $D \times E + B \times H$ acting upon ω . If Heaviside’s analysis is correct, then this must be the same torque implied by the stress given by equation (50). And indeed it is, as we can see by substitution in our expression for $\tau \times e_n$.

The rather stringent limitation to rigid-body transport deprives Heaviside’s result of general significance, but that did not bother him because he was here concerned with situations in which the velocities of the sources are small compared with the velocity of radiation. He remarked [Heaviside 1950 (1893): sec. 124]:

Should, however, the velocities of the connected sources be only a very small fraction of the speed of propagation of disturbances through the medium, the re-adjustment of the displacement as the sources move takes place practically instantaneously (as if the speed of propagation were infinite), so that the displacement system remains unchanged, preserving its stationary type, whilst it moves through the medium as a rigid body, in rigid connection with the sources. (This is obviously quite incorrect at a great distance from the sources, but there the effects themselves are insensible.)

12. Thomson’s unusual vigor in applying Lagrangian techniques can be gauged from the following famous remark attributed to W. G. Clay, ninth Wrangler in 1887:

You have applied Dynamics to Physics and Chemistry; why not extend the application still further? This is the task I propose to you – to find the Lagrangian Function of the Universe. Think of the advantage of being able to obtain by a simple mathematical process the report of the Syndicates on any matter, for such things, I imagine, would correspond to Equations of Motion.

From Rayleigh, 1969 (1942): 46.

13. Though he did apparently develop a theory of “energy tubes” radiated by charges which he hoped to use to unify gravitation with electromagnetism. This work was to have appeared as volume 4 of the *Electromagnetic Theory*, but the MS was lost shortly after his death; only scattered notes for it remain. See Josephs, 1950 for an account of what the lost volume might have contained. This work, while novel, seems entirely in keeping with Heaviside’s Maxwellianism, only importing into it statistical considerations derived from the properties of “energy tubes”.